A first course of homological algebra

D. G. NORTHCOTT

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CONTENTS

PREFACE

The main part of this book is an expanded version of lectures which I gave at Sheffield University during the session 1971–2. These lectures were intended to provide a first course of Homological Algebra, assuming only a knowledge of the most elementary parts of the theory of modules. The amount of time available was very limited and ruled out any approach which required the elaborate machinery or great generality that is sometimes associated with the subject. The alternative, it seemed to me, was to build the course round a number of topics which I hoped my audience would find interesting, and create the necessary tools by *ad hoc* constructions. Fortunately it proved rather easy to find topics where the techniques needed to treat one of them could also be used on the others. In the event, the first five chapters were fully covered in the course. The last chapter was added later and it differs from those that precede it by including some material which, so far as I am aware, has not previously appeared in print. This material has to do with what are here called *semi-commutative local algebras*. It is hoped that it may be of some interest to the specialist as well as to the beginner.

Reference has already been made to one way in which the amount of available time influenced the structure of the course. It had, indeed, a second effect. In order to speed up the presentation, some easily proved results and parts of some demonstrations were left as exercises. Other exercises were included in order to expand the main themes. What actually happened was that two members of the class, Mr A. S. McKerrow and Mr P. M. Scott, were good-natured enough to do all the exercises and, in addition, they provided the other participants with copies of their solutions. These solutions, edited so as to remove differences of style, are reproduced here. However the reader will find that his grasp of the subject is much improved if he works out a fair proportion of the problems for himself, rather than merely checks through the details of the arguments provided. The more difficult exercises have been marked with an asterisk.

I am much indebted to other mathematicians who have written on similar or related topics, and the list of references at the end shows the books and papers that I have consulted recently. It is a pleasure to acknowledge the help and benefit that I have derived from these and other sources. I have not attempted to compile a comprehensive bibliography. Naturally the degree of my indebtedness varies from one author to another. I have, for example, made much use of I. Kaplansky's treatment of homological dimension. Also I am very conscious of the influence which the writings of H. Bass and R. G. Swan have had on this account.

As on other occasions, I have been very fortunate in the help that has been given to me. Once again my secretary, Mrs E. Benson, has converted pages of untidy manuscript into an orderly form where the idea that they might turn into a book no longer seemed unreasonable. Besides this Mr A. S. McKerrow checked much of the first draft to see that it was technically correct. Their assistance has been extremely valuable and I am most grateful to them both.

D. G. NORTHCOTT

Sheffield
October 1972

NOTES FOR THE READER

This opportunity is taken to summarize what the reader is assumed to know already, and to draw his attention to any conventions or terminology which may differ slightly from those to which he has been accustomed.

All the main topics in this book have to do with *rings* and *modules*. First a word about rings. Unless otherwise stated, these need not be commutative, but every one is required to have an identity element. (Usually the identity element does not have to be different from the zero element.) When we speak of a homomorphism of one ring into another, it is to be understood that the identity element of the former is mapped into that of the latter. In particular, if Γ is a *subring* of a ring Λ, that is if the inclusion mapping $\Gamma \to \Lambda$ is a ring-homomorphism, then our convention ensures that Γ and Λ must have the same identity element. An important subring of Λ is its *centre*. This, of course, is composed of all elements γ with the property that $\lambda\gamma = \gamma\lambda$ for every λ in Λ.

Let Λ be a ring. In any reference to a Λ-*module* it is always intended that multiplication of an element of the module by the identity 1_Λ, of Λ, shall leave the element of the module unchanged. In other words, we only consider *unitary* modules. Note that there are two *types* of Λ-module, namely *left* Λ-modules and *right* Λ-modules.† The system formed by all left resp. right Λ-modules (and the homomorphisms between them) is referred to as the *category* of left resp. right Λ-modules and is denoted by $\mathscr{C}^L_\Lambda$ resp. $\mathscr{C}^R_\Lambda$. Though use is made of the language of Category Theory it is not at all necessary that the reader should have previously met the definition of an abstract category. To illustrate the language let us observe that a module over the ring Z of integers is just the same as an (additively written) abelian group. Further if A and B are two such objects, then a mapping $f: A \to B$ is a homomorphism of Z-modules if and only if it is a group-homomorphism. A convenient way in which to describe all this is to say that *the category of Z-modules can be identified with the category of (additively written) abelian groups.*

Although we assume no general knowledge of Category Theory it is

† If the ring is *commutative* we do not need to make this distinction.

supposed that the reader is familiar with the elementary theory of modules and, on this basis, certain terms are used without explanation. The following are typical examples: *submodule*, *factor module*; *image*; *kernel* and *cokernel* (of a homomorphism); *exact sequence*, *commutative diagram*; *direct sum* and *direct product*. In addition we take as known the standard *isomorphism theorems* and presuppose some elementary knowledge of transfinite methods based on well-ordering and *Zorn's Lemma*. A leisurely account of these matters will be found in (20) in the list of references, should the reader wish to supplement his knowledge.

Let $f: A \to B$ be a homomorphism of Λ-modules. If, in addition, f is an injective mapping, then, of course, it is customary to say that f is a *monomorphism*. We shall also say that f is *monic* whenever we wish to describe a situation of this kind. This is done solely to expand a limited vocabulary which otherwise could lead to tedious repetition. For the same reason, if the homomorphism f is a surjective mapping, then we shall say either that f is an *epimorphism* or that it is *epic* depending on which alternative description happens to be the more convenient.

Our next remarks concern notation in relation to sets and modules. Thus if A is a set, then i_A always denotes the *identity mapping* of A. Now suppose that X and Y are sets. If X is a subset of Y and we wish to indicate this, then we shall write $X \subseteq Y$. However, if X is a *proper subset* of Y, that is if $X \subseteq Y$ but $X \neq Y$, then $X \subset Y$ will be used to convey this information.

Turning now to modules, let Λ be a ring and $\{A_i\}_{i \in I}$ a family of Λ-modules. The family will have both a direct sum and a direct product. The former of these will be denoted by $\bigoplus_{i \in I} A_i$ and the latter by $\prod_{i \in I} A_i$. However when we have to do with a finite family

$$\{A_1, A_2, \ldots, A_n\},$$

then we use $A_1 \oplus A_2 \oplus \ldots \oplus A_n$ and $A_1 \times A_2 \times \ldots \times A_n$ as alternatives to $\bigoplus_{i=1}^{n} A_i$ and $\prod_{i=1}^{n} A_i$ respectively. Again if A is a Λ-module, then $\bigoplus_{i \in I} A$ or $\bigoplus_{I} A$ will denote a direct sum in which all the summands are equal to A and there is one of them for each member of I. Likewise $\prod_{i \in I} A$ or $\prod_{I} A$ will denote a direct product in which each factor is A and there is one factor for each element of the set I.

It is hoped that enough has now been said to prepare the reader. Note that the numbering of theorems, lemmas and so on is begun afresh in each chapter. If a reference is made to a result and no chapter or section is specified, then the result in question is to be found in the chapter being read. In all other cases the extra information needed for identification is provided.

1

THE LANGUAGE OF FUNCTORS

1.1 Notation

Λ, Γ, Δ will denote rings with identity elements. They need not be commutative. Z will denote the ring of integers. The category of left (resp. right) Λ-modules will be denoted by $\mathscr{C}_\Lambda^L$ (resp. $\mathscr{C}_\Lambda^R$). Sometimes it is immaterial whether we work exclusively with left Λ-modules or exclusively with right Λ-modules. In such a case $\mathscr{C}_\Lambda$ will denote the category in question. When Λ is commutative, we make no distinction between $\mathscr{C}_\Lambda^L$ and $\mathscr{C}_\Lambda^R$. Also we normally identify the category of additively written abelian groups with the category of Z-modules. Finally i_A is used to denote the identity map of A.

1.2 Bimodules

Suppose that A is both a Λ-module and a Γ-module, the additive structure being the same in both cases. Let us suppose that multiplication (of an element of A) by an element of Λ always commutes with multiplication by an element of Γ. We then say that A is a (Λ, Γ)-*bimodule*. If, for example, Λ operates on the left and Γ on the right, we may indicate this by writing ${}_\Lambda A_\Gamma$. If A and A' are both (Λ, Γ)-bimodules of the same type, then a mapping $f: A \to A'$ which is simultaneously Λ-linear and Γ-linear is called a *bihomomorphism*.

***Example* 1.** Every Λ-module is a (Λ, Z)-bimodule.

***Example* 2.** If Γ is the centre of Λ, then every Λ-module is a (Λ, Γ)-bimodule.

***Example* 3.** Λ itself is a (Λ, Λ)-bimodule with one Λ acting on the right and the other on the left. This is by virtue of the associative law of multiplication.

1.3 Covariant functors

Suppose that with each module A in $\mathscr{C}_\Lambda$ there is associated a module $F(A)$ in $\mathscr{C}_\Delta$ and that to each Λ-homomorphism $f: A \to A'$ there corresponds a Δ-homomorphism $F(f): F(A) \to F(A')$. Suppose further that

(1) $F(i_A) = i_{F(A)}$ *for all* A *in* $\mathscr{C}_\Lambda$;

(2) $F(gf) = F(g)F(f)$ *whenever* $f: A \to A'$ *and* $g: A' \to A''$ *in* $\mathscr{C}_\Lambda$.

In these circumstances we say we have a *covariant functor* $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ from Λ-modules to Δ-modules. Simple commutative diagrams (of Λ-modules and Λ-homomorphisms) such as

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ A' \arrow[rr] & & A'' \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A_1 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_3 \arrow[r] & A_4 \end{tikzcd}$$

remain commutative when a covariant functor is applied. Also if $f: A \to A'$ is an isomorphism and $g: A' \to A$ is its inverse, then, for a covariant functor F, $F(f): F(A) \to F(A')$ is an isomorphism and $F(g): F(A') \to F(A)$ is its inverse. This is because gf and fg are identity maps.

For the remainder of section (1.3), $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ will denote a covariant functor.

Definition. *F is said to be 'additive' if whenever $f_1: A \to A'$ and $f_2: A \to A'$ are Λ-homomorphisms, sharing a common domain A and a common codomain A', we have $F(f_1 + f_2) = F(f_1) + F(f_2)$.*

Note. The Λ-homomorphisms of A into A' form an abelian group. This is denoted by $\operatorname{Hom}_\Lambda(A, A')$. Addition in $\operatorname{Hom}_\Lambda(A, A')$ is defined by $(f_1 + f_2)(a) = f_1(a) + f_2(a)$.

If F is additive, then it carries null homomorphisms and null modules into null homomorphisms and null modules.

In the classical theory of modules, *finite* direct sums and *finite* direct products are indistinguishable. Here this is recognized by introducing the notion of a *biproduct.*

Let $A_1, A_2, \ldots, A_n$ and A be Λ-modules and suppose we are given homomorphisms $\sigma_i: A_i \to A\,(1 \leqslant i \leqslant n)$ and $\pi_i: A \to A_i\,(1 \leqslant i \leqslant n)$. The complete system is called a representation of A as a *biproduct* of $A_1, A_2, \ldots, A_n$ if

(*a*) $\pi_j \sigma_i = \delta_{ji}$, i.e. $\pi_j \sigma_i$ is a null resp. identity homomorphism if $i \neq j$ resp. $i = j$;

(*b*) $\Sigma \sigma_i \pi_i = $ identity.

In these circumstances we write variously

$$A = A_1 \oplus A_2 \oplus \ldots \oplus A_n \text{ (direct sum notation)},$$
$$A = A_1 \times A_2 \times \ldots \times A_n \text{ (direct product notation)},$$
$$A = A_1 * A_2 * \ldots * A_n \text{ (biproduct notation)},$$

and, more explicitly,

$$[\sigma_1, \ldots, \sigma_n; A; \pi_1, \ldots, \pi_n] = A_1 * A_2 * \ldots * A_n.$$

We call $\sigma_i: A_i \to A$ the *canonical injection* (it is necessarily a monomorphism) and $\pi_i: A \to A_i$ the *canonical projection* (it is necessarily an epimorphism).

Exercise 1.† *Let* $[\sigma_1, \ldots, \sigma_n; A; \pi_1, \ldots, \pi_n] = A_1 * A_2 * \ldots * A_n$ *in* $\mathscr{C}_\Lambda$. *Show that if* Λ*-homomorphisms* $f_i: A_i \to B$ $(1 \leqslant i \leqslant n)$ *are given, then there exists a unique homomorphism* $f: A \to B$ *such that* $f\sigma_i = f_i$ *for* $1 \leqslant i \leqslant n$. *Show also that if* $g_i: B \to A_i$ $(1 \leqslant i \leqslant n)$ *are prescribed* Λ*-homomorphisms, then there exists a unique homomorphism* $g: B \to A$ *such that* $\pi_i g = g_i$ *for* $1 \leqslant i \leqslant n$.

Exercise 2. *Let* $[\sigma_1, \ldots, \sigma_n; A; \pi_1, \ldots, \pi_n] = A_1 * A_2 * \ldots * A_n$ *in* $\mathscr{C}_\Lambda$. *Show that the homomorphism* $A_1 \oplus A_2 \oplus \ldots \oplus A_n \to A$ *induced by the* σ_i *and the homomorphism* $A \to A_1 \times A_2 \times \ldots \times A_n$ *induced by the* π_i *are both of them isomorphisms.*

Observe that if $A_1, A_2, \ldots, A_n$ are given, then we can always find $A, \sigma_1, \sigma_2, \ldots, \sigma_n$ and $\pi_1, \pi_2, \ldots, \pi_n$ so that

$$[\sigma_1, \ldots, \sigma_n; A; \pi_1, \ldots, \pi_n] = A_1 * A_2 * \ldots * A_n.$$

Theorem 1. Let $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ *be an additive covariant functor and let* $[\sigma_1, \ldots, \sigma_n; A; \pi_1, \ldots, \pi_n] = A_1 * A_2 * \ldots * A_n$ *in* $\mathscr{C}_\Lambda$. *Then*
$[F(\sigma_1), \ldots, F(\sigma_n); F(A); F(\pi_1), \ldots, F(\pi_n)] = F(A_1) * F(A_2) * \ldots * F(A_n)$ *in* $\mathscr{C}_\Delta$.

Proof. Apply F to the relations $\pi_j \sigma_i = \delta_{ji}$ and $\Sigma\, \sigma_i \pi_i =$ identity.

We shall now show that this property characterizes additive covariant functors.

Theorem 2. *Let* $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ *be a covariant functor and suppose that whenever* $[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2$ *in* $\mathscr{C}_\Lambda$, *then*

$$[F(\sigma_1), F(\sigma_2); F(A); F(\pi_1), F(\pi_2)] = F(A_1) * F(A_2) \text{ in } \mathscr{C}_\Delta.$$

In these circumstances F is additive.

† Solutions to the Exercises will be found at the end of the chapter.

Proof. Let $f_1, f_2 : A \to B$ be homomorphisms. Further, let

$$[\sigma_1, \sigma_2; C; \pi_1, \pi_2] = A * A.$$

Then $[F(\sigma_1), F(\sigma_2); F(C); F(\pi_1), F(\pi_2)] = F(A) * F(A)$ and therefore $i_{F(C)} = F(\sigma_1) F(\pi_1) + F(\sigma_2) F(\pi_2)$. Define $d : A \to C$ by $d = \sigma_1 + \sigma_2$. Then $\pi_1 d = \pi_1(\sigma_1 + \sigma_2) = i_A$ from which we obtain

$$F(\pi_1) F(d) = F(\pi_1 d) = F(i_A) = i_{F(A)}.$$

Similarly $F(\pi_2) F(d) = i_{F(A)}$. Now

$$F(d) = i_{F(C)} F(d) = (F(\sigma_1) F(\pi_1) + F(\sigma_2) F(\pi_2)) F(d).$$

Hence

$$\begin{aligned} F(d) &= F(\sigma_1) F(\pi_1) F(d) + F(\sigma_2) F(\pi_2) F(d) \\ &= F(\sigma_1) i_{F(A)} + F(\sigma_2) i_{F(A)} = F(\sigma_1) + F(\sigma_2). \end{aligned}$$

Define $g : C \to B$ by $g = f_1 \pi_1 + f_2 \pi_2$. Then

$$g\sigma_1 = (f_1 \pi_1 + f_2 \pi_2) \sigma_1 = f_1 \pi_1 \sigma_1 + f_2 \pi_2 \sigma_1 = f_1.$$

Similarly $g\sigma_2 = f_2$. Furthermore $gd = (f_1 \pi_1 + f_2 \pi_2)(\sigma_1 + \sigma_2) = f_1 + f_2$. Accordingly $F(f_1 + f_2) = F(gd) = F(g) F(d) = F(g)(F(\sigma_1) + F(\sigma_2))$. Thus

$$\begin{aligned} F(f_1 + f_2) &= F(g) F(\sigma_1) + F(g) F(\sigma_2) \\ &= F(g\sigma_1) + F(g\sigma_2) \\ &= F(f_1) + F(f_2). \end{aligned}$$

Hence f is additive.

Theorem 3. *Suppose that* $[\sigma_1, \sigma_2; A; \pi_1, \pi_2) = A_1 * A_2$ *in* $\mathscr{C}_\Lambda$. *Then the sequences*

$$0 \to A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \to 0 \tag{1.3.1}$$

and

$$0 \to A_2 \xrightarrow{\sigma_2} A \xrightarrow{\pi_1} A_1 \to 0 \tag{1.3.2}$$

are exact.

Proof. We need only consider (1.3.1) and for this it suffices to show $\operatorname{Ker} \pi_2 \subseteq \operatorname{Im} \sigma_1$. Let $\alpha \in \operatorname{Ker} \pi_2$. Then

$$\alpha = \sigma_1 \pi_1(\alpha) + \sigma_2 \pi_2(\alpha) = \sigma_1 \pi_1(\alpha) \in \operatorname{Im} \sigma_1.$$

Lemma 1. *Suppose that* $A_1 \xrightarrow{\sigma_1} A$ *and* $A \xrightarrow{\pi_1} A_1$ *are* Λ*-homomorphisms such that* $\pi_1 \sigma_1 =$ *identity. Then* $A = \operatorname{Im} \sigma_1 \oplus \operatorname{Ker} \pi_1$.

Proof. Let $a \in A$. Then $\pi_1(a - \sigma_1 \pi_1(a)) = 0$ and therefore

$$a = \sigma_1 \pi_1(a) + (a - \sigma_1 \pi_1(a)) \in \operatorname{Im} \sigma_1 + \operatorname{Ker} \pi_1.$$

Now assume that $\alpha \in \operatorname{Im}\sigma_1 \cap \operatorname{Ker}\pi_1$, say $\alpha = \sigma_1(a_1)$ with $a_1 \in A_1$. Then

$$a_1 = \pi_1\sigma_1(a_1) = \pi_1(\alpha) = 0$$

whence $\alpha = 0$. This shows that $A = \operatorname{Im}\sigma_1 \oplus \operatorname{Ker}\pi_1$.

Theorem 4. *Let $0 \to A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \to 0$ be an exact sequence in $\mathscr{C}_\Lambda$. Then the following statements are equivalent:*

(1) $\operatorname{Im}\sigma_1\,(= \operatorname{Ker}\pi_2)$ *is a direct summand of* A;

(2) *there exists a Λ-homomorphism $\pi_1: A \to A_1$ such that* $\pi_1\sigma_1 = identity$;

(3) *there exists a Λ-homomorphism $\sigma_2: A_2 \to A$ such that* $\pi_2\sigma_2 = identity$;

(4) *there exist Λ-homomorphisms $\sigma_2: A_2 \to A$ and $\pi_1: A \to A_1$ such that* $[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2$.

Proof. By the definitions and Lemma 1,

$$(4) \Rightarrow (2) \Rightarrow (1) \quad \text{and} \quad (4) \Rightarrow (3) \Rightarrow (1).$$

Assume (1), say $A = \operatorname{Im}\sigma_1 \oplus B$ for some submodule B of A. Now σ_1 induces an isomorphism $A_1 \xrightarrow{\sim} \operatorname{Im}\sigma_1$. Let $u: \operatorname{Im}\sigma_1 \xrightarrow{\sim} A_1$ be its inverse. Next π_2 induces an isomorphism $B \xrightarrow{\sim} A_2$. Let $v: A_2 \xrightarrow{\sim} B$ be its inverse. Put $\pi_1 = up$ and $\sigma_2 = jv$, where $p: A \to \operatorname{Im}\sigma_1$ is the projection associated with the relation $A = \operatorname{Im}\sigma_1 \oplus B$ and $j: B \to A$ is an inclusion mapping. Then $\pi_1\sigma_1 = \text{identity}$, $\pi_1\sigma_2 = 0$, $\pi_2\sigma_1 = 0$, $\pi_2\sigma_2 = \text{identity}$. Finally if $a \in A$, then $\sigma_1\pi_1(a)$ is the projection of a on $\operatorname{Im}\sigma_1$ and $\sigma_2\pi_2(a)$ is the projection of a on B. Thus

$$\sigma_1\pi_1(a) + \sigma_2\pi_2(a) = a \quad \text{or} \quad \sigma_1\pi_1 + \sigma_2\pi_2 = \text{identity}.$$

Accordingly (1) implies (4).

Definition. *Let $0 \to A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \to 0$ be an exact sequence in $\mathscr{C}_\Lambda$. If the four equivalent conditions of Theorem 4 hold, then it is called a 'split exact sequence'.*

We now see, in view of Theorem 3, that if $[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2$, then

$$0 \to A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \to 0$$

and

$$0 \to A_2 \xrightarrow{\sigma_2} A \xrightarrow{\pi_1} A_1 \to 0$$

are split exact sequences. On the other hand, if $0 \to B \to A \to C \to 0$ is a split exact sequence, then we have an isomorphism $A \approx B \oplus C$.

Theorem 5. *Let* $0 \to A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \to 0$ *be a split exact sequence in* $\mathscr{C}_\Lambda$ *and* $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ *an additive covariant functor. Then*

$$0 \to F(A_1) \xrightarrow{F(\sigma_1)} F(A) \xrightarrow{F(\pi_2)} F(A_2) \to 0$$

is a split exact sequence in $\mathscr{C}_\Delta$.

Proof. Choose $\sigma_2: A_2 \to A$, $\pi_1: A \to A_1$ so that

$$[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2.$$

Then, by Theorem 1, $[F(\sigma_1), F(\sigma_2); F(A); F(\pi_1), F(\pi_2)] = F(A_1) * F(A_2)$ and therefore

$$0 \to F(A_1) \xrightarrow{F(\sigma_1)} F(A) \xrightarrow{F(\pi_2)} F(A_2) \to 0$$

is a split exact sequence by virtue of Theorem 4.

Exercise 3. *In the diagram*

$$\begin{tikzcd}
A_1 \arrow[dr, "\sigma_1"] & & A_1 \\
& A \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
A_2 \arrow[ur, "\sigma_2"] & & A_2
\end{tikzcd}$$

suppose that $\pi_1 \sigma_1 =$ *identity and that* $\pi_2 \sigma_2 =$ *identity. Suppose also that*

$$A_1 \xrightarrow{\sigma_1} A \xrightarrow{\pi_2} A_2 \quad \text{and} \quad A_2 \xrightarrow{\sigma_2} A \xrightarrow{\pi_1} A_1$$

are exact. Show that $[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2$.

Exercise 4. *Let* $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ *be a covariant functor and suppose that whenever* $0 \to A' \to A \to A'' \to 0$ *is a split exact sequence in* $\mathscr{C}_\Lambda$, *then* $0 \to F(A') \to F(A) \to F(A'') \to 0$ *is a split exact sequence in* $\mathscr{C}_\Delta$. *Deduce that* F *is additive.*

Let $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ be a covariant functor. Assume that whenever $0 \to A' \to A \to A'' \to 0$ is exact in $\mathscr{C}_\Lambda$, then

$$0 \to F(A') \to F(A) \to F(A'')$$

resp. $F(A') \to F(A) \to F(A'') \to 0$

is exact in $\mathscr{C}_\Delta$. In these circumstances we say that F is *left exact* resp. *right exact*. Should it be the case that the exactness of

$$0 \to A' \to A \to A'' \to 0$$

only implies that of $\quad F(A') \to F(A) \to F(A''),$

then F is said to be *half exact*. If F is both left and right exact, i.e. if $0 \to A' \to A \to A'' \to 0$ is exact always implies that

$$0 \to F(A') \to F(A) \to F(A'') \to 0$$

is exact, then F is said to be an *exact* functor.

Let $F: \mathscr{C}_\Lambda \to \mathscr{C}_\Delta$ be a covariant functor. If F is left exact then it preserves monomorphisms, whereas if it is right exact it preserves epimorphisms.

Lemma 2. *Suppose that the covariant functor F is left exact and that $0 \to A_1 \to A \to A_2$ is exact in $\mathscr{C}_\Lambda$. Then $0 \to F(A_1) \to F(A) \to F(A_2)$ is exact in $\mathscr{C}_\Delta$.*

The proofs of this and the next two lemmas are straightforward and will be omitted. In both Lemmas 3 and 4, F is understood to be a *covariant* functor from $\mathscr{C}_\Lambda$ to $\mathscr{C}_\Delta$.

Lemma 3. *Suppose that F is right exact and $A_1 \to A \to A_2 \to 0$ is exact in $\mathscr{C}_\Lambda$. Then $F(A_1) \to F(A) \to F(A_2) \to 0$ is exact in $\mathscr{C}_\Delta$.*

Lemma 4. *Suppose that F is exact and $A_1 \to A \to A_2$ is an exact sequence in $\mathscr{C}_\Lambda$. Then $F(A_1) \to F(A) \to F(A_2)$ is exact in $\mathscr{C}_\Delta$.*

Theorem 6. *If the covariant functor F is half exact, then it is additive.*

Proof. Let $[\sigma_1, \sigma_2; A; \pi_1, \pi_2] = A_1 * A_2$ in $\mathscr{C}_\Lambda$. By Theorem 2, it is enough to show that $[F(\sigma_1), F(\sigma_2); F(A); F(\pi_1), F(\pi_2)]$ equals

$$F(A_1) * F(A_2).$$

Now by Theorem 3 and the half exactness of F,

$$F(A_1) \xrightarrow{F(\sigma_1)} F(A) \xrightarrow{F(\pi_2)} F(A_2)$$

and $$F(A_2) \xrightarrow{F(\sigma_2)} F(A) \xrightarrow{F(\pi_1)} F(A_1)$$